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Fundamental solution for bonded materials with a free surface parallel to the interface. Part I: Solution of concentrated forces acting at the inside of the material with a free surface

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Abstract

The problem studied in this paper is that of a coated semi-infinite plane subjected to a concentrated force in the upper thin layer (or film). The elastic properties of the coating material are different from those of the substrate, and a perfect bond is assumed between the two materials. The exact solutions of stress functions in a series form are obtained by the method of image. The terms in series form of the stress functions correspond to the image points from the lower order to the higher. The recurrence relations of the stress functions are given, i.e., the stress functions corresponding to the higher order image points are determined by the lower ones. Hence, from the original stress functions for an infinite plane subjected to a concentrated force, the explicit formulas of all terms of the stress function series can be derived. Also, through comparisons between the theoretical results and the numerical results by FEM, it is verified that the convergence rate of the solutions is very rapid. In most practical cases only the first several image points are sufficient to ensure the accuracy of the solutions.

Keywords: Image point method; Stress function; Concentrated force; Fundamental solution; Dissimilar material; Film

1. Introduction

In order to improve the surface properties, modern material technology very often requires the substrate material to be coated with one thin film or layer. Due to the extensive applications of the film technology in industries, the studies of interface strength about this kind of bimaterial have been of great scientific

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interests. Varias et al. (1999) simulated the transient problem of film buckling and interfacial failure using a numerical method. Balkan and Madenci (1998) analyzed the response of a thin film over a substrate with circular debonding under thermal loading. Wei and Hutchinson (1998) proposed and analyzed a cohesive zone model for steady-state peeling of thin, rate-independent, and elastic–plastic film bonded to an elastic substrate.

In this paper, we model the surface-modified material as a bonded dissimilar material with a free surface, and focus on the theoretical solutions for concentrated forces in the surface layer (or film) through the image method, based on the previous work done by Xu et al. (2002), in which the theoretical solution of a concentrated force on the free surface of coating material was obtained by the image method.

It can be found that the idea of the image method to deal with the solid mechanics problem has been reported in early times, starting from the problem of infinite isotropic body subjected to a concentrated force solved by Timoshinko and Goodier (1987). For bonded dissimilar materials, the method of image has also been used. Dundurs and Hetényi (1961), Hetényi and Dundurs (1962) obtained the closed-form solutions of the elastic plane with a circular inserted by this method in terms of Ariy stress functions for an infinite plane with a point force. In recent years, a number of works on bimaterials have been done to analyze the stress fields through the image method. Ting (1992) presented simple explicit expressions of Green's functions for anisotropic elastic half-spaces and bimaterials subjected to line forces and line dislocations, and discussed the image singularities of the Green's functions. Aderogba (2000, 2003) established a theorem for generating the Airy stress function for trimaterial due to a point force utilizing the method of image. Ma and Lin (2001) found that the fundamental solutions required to construct all the image singularities of applied forces and dislocations for the half-space are only forces and dislocations and their differentiations in the infinite space. Wu et al. (2002) obtained the exact solutions for interfacial edge dislocations in an anisotropic bicrystal under plane strain by the method of image dislocations.

In the present study, due to the free surface and interface, an infinite series of images are produced from the point where the load is applied. The two dimensional solution is deduced in detail by using the infinite series of the Goursat's stress functions corresponding to each image point. It is found that the stress functions corresponding to higher order image points can be determined from those corresponding to lower ones, therefore, all of the stress functions can be determined starting from the stress functions corresponding to the first order image point which is in fact the stress functions for a infinite plane subjected to a concentrated force. Also, it is verified that the first several image points have major influence on the accuracy of the theoretical solution for most practical cases of materials combination.

2. Analytical model

A bonded dissimilar material with a free surface is considered, as shown in Fig. 1(a). The point force applies in the upper thin layer (or film), denoted by material 'I', of which the thickness is 'h', shear modulus μ_1, and Poisson's ratio v_1. The substrate denoted by material 'II' is a half-infinite plane, of which the shear modulus is μ_2 and Poisson's ratio v_2. The global coordinate is set as originating at the load point.

It is convenient to classify the image points into two series. First series of the image points are shown as Fig. 1(b). Firstly, an image point is produced by the free surface and then reflected by the interface, and so on, resulting in a series of infinite image points. The second series of infinite image points are shown as Fig. 1(c). Firstly, an image point is produced by the interface and then reflected by the free surface, and so on, also resulting in a series of infinite image points.

In Fig. 1(b), the image points above the free surface are denoted by O_i, and the corresponding local coordinates are expressed by complex form as $z_k = x + iy_{k1}$. The image points beneath the interface are denoted by C_i , the corresponding local coordinates: $\varsigma_k = x + iy_{k2}$. The relationship between the local coordinates and the global coordinates are:

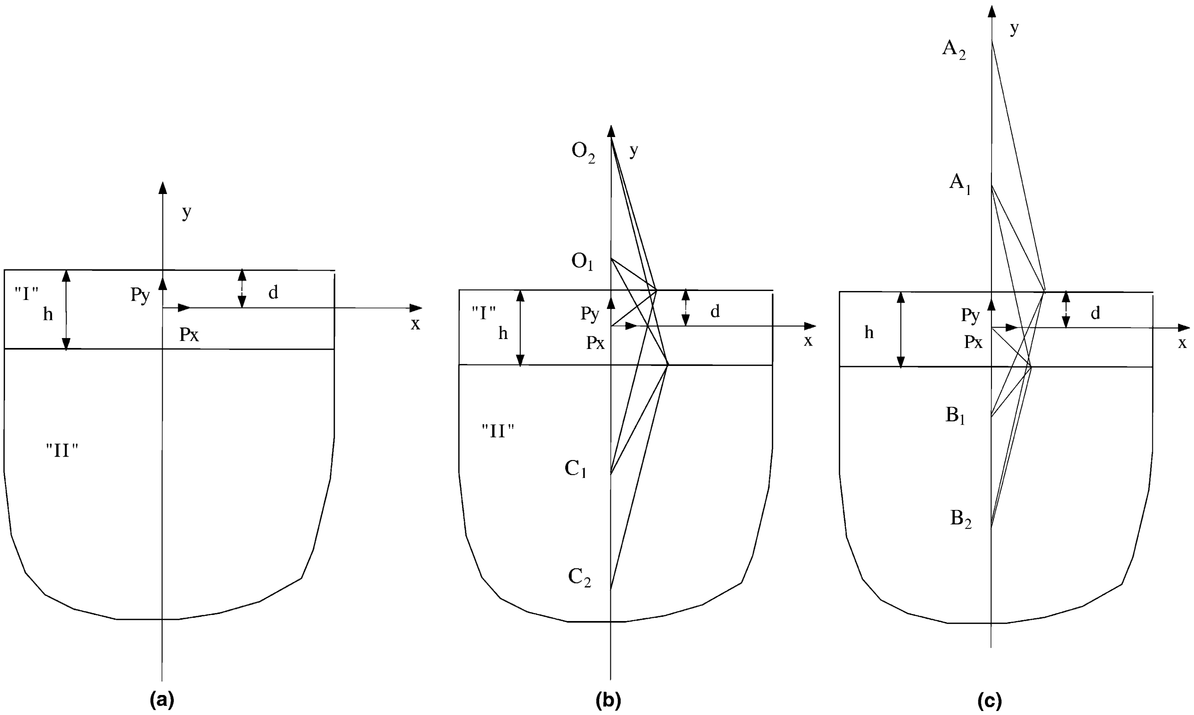

Fig. 1. (a) Analytical model; (b) first series of image points; (c) second series of image points.

$$z = x + \mathrm{i}y, \qquad \bar{z} = x - \mathrm{i}y, \quad \mathrm{i} = \sqrt{-1}$$
$$z_k = z - [2d + 2(k-1)h]\mathrm{i} \tag{1}$$
$$\varsigma_k = z + 2kh\mathrm{i}$$

In Fig. 1(c), image points above the free surface are denoted by A_i, and local coordinates: $\eta_k = x + \mathrm{i}y_{k3}$. The image points beneath the interface are denoted by B_i, the corresponding local coordinates: $\xi_k = x + \mathrm{i}y_{k4}$. The relationship between the local coordinates and the global coordinates can be expressed as:

$$\begin{aligned} \xi_k &= z + (2kh - 2d)\mathrm{i} \\ \eta_k &= z - 2kh\mathrm{i} \end{aligned} \tag{2}$$

The continuity conditions at the interface are:

$$\left.\begin{aligned} \sigma_{y\mathrm{I}} + \mathrm{i}\tau_{xy\mathrm{I}} &= \sigma_{y\mathrm{II}} + i\tau_{xy\mathrm{II}} \\ u_\mathrm{I} + \mathrm{i}v_\mathrm{I} &= u_\mathrm{II} + \mathrm{i}v_\mathrm{II} \end{aligned}\right. \quad \text{at } y = -(h - d) \tag{3}$$

The boundary condition of tractions free at the free surface can be written as:

$$\sigma_{y\mathrm{I}} + \mathrm{i}\tau_{xy\mathrm{I}} = 0 \quad \text{at } y = d \tag{4}$$

3. Recurrence relationships of stress functions

3.1. Goursat stress function

Goursat complex stress functions for plane problem can be expressed as:

$$\begin{aligned} &\sigma_y + \mathrm{i}\tau_{xy} = \varphi' + \bar{\varphi}' + \bar{z}\varphi'' + \psi', \quad \sigma_x + \sigma_y = 4Re(\varphi') \\ &2\mu(u + iv) = \kappa\varphi - z\bar{\varphi}' - \overline{\psi} \end{aligned} \tag{5}$$

where

$$\begin{cases} \kappa = 3 - 4v & \text{for plane strain} \\ \kappa = (3 - v)/(1 + v) & \text{for plane stress} \end{cases} \tag{6}$$

From the variable relations in Eqs. (1) and (2) one can get:

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}, \quad \dfrac{\partial}{\partial y} = \mathrm{i}\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right) \\ \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - \mathrm{i}\dfrac{\partial}{\partial y}\right), \quad \dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + \mathrm{i}\dfrac{\partial}{\partial y}\right) \\ \dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z_k}, \quad \dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial \varsigma_k}, \dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial \eta_k}, \quad \dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial \xi_k} \end{cases} \tag{7}$$

Set the stress functions as the following:

$$\begin{cases} \varphi_{\mathrm{I}} = F_{\mathrm{I}}(z) + \sum_{k=1}^{\infty}[A_k(z_k) + B_k(\zeta_k) + C_k(\xi_k) + D_k(\eta_k)] \\ \psi_{\mathrm{I}} = f_{\mathrm{I}}(z) + \sum_{k=1}^{\infty}[a_k(z_k) + b_k(\zeta_k) + c_k(\xi_k) + d_k(\eta_k)] \end{cases} \tag{8}$$

$$\begin{cases} \varphi_{\mathrm{II}} = F_{\mathrm{II}}(z) + \sum_{k=1}^{\infty}[F_k(z_k) + G_k(\eta_k)] \\ \psi_{\mathrm{II}} = f_{\mathrm{II}}(z) + \sum_{k=1}^{\infty}[f_k(z_k) + g_k(\eta_k)] \end{cases} \tag{9}$$

Here, $F_{\mathrm{I}}, f_{\mathrm{I}}, F_{\mathrm{II}}, f_{\mathrm{II}}, A_k$—$D_k$ and a_k—d_k, are all complex-variable analytic functions. The functions φ_{I}, ψ_{I}, φ_{II} and ψ_{II} correspond to materials ‘I’, ‘II’, respectively.

3.2. Stress functions recurrence for image points symmetric to the interface

At the interface:

$$z = x - (h - d)\mathrm{i}, \quad z_k = \bar{\varsigma}_k, \quad \xi_{k+1} = \bar{\eta}_k \tag{10}$$

Substituting the stress functions expressed by Eqs. (8) and (9) into the continuity conditions of Eq. (3) and considering the corresponding relationships of the image points yields

$$\begin{aligned} &F_I' + \overline{F_{\mathrm{I}}'} + [x + (h-d)\mathrm{i}]F_{\mathrm{I}}'' + f_{\mathrm{I}}' + C_1' + \overline{C_1'} + [x + (h-d)\mathrm{i}]C_1'' + c_1' = F_{\mathrm{II}}' + \overline{F_{\mathrm{II}}'} + [x + (h-d)\mathrm{i}]F_{\mathrm{II}} + f_{\mathrm{II}}' \\ &A_k' + B_k' + \overline{A_k'} + \overline{B_k'} + [x + (h-d)\mathrm{i}][A_k'' + B_k''] + a_k' + b_k' = F_k' + \overline{F_k'} + [x + (h-d)\mathrm{i}]F_k'' + f_k' \\ &C_{k+1}' + D_k' + \overline{C_{k+1}'} + \overline{D_k'} + [x + (h-d)\mathrm{i}][C_{k+1}'' + D_k''] + c_{k+1}' + d_k' = G_k' + \overline{G_k'} + [x + (h-d)\mathrm{i}]G_k'' + g_k' \\ &\Gamma\left\{\kappa_1 F_I - [x - (h-d)\mathrm{i}]\overline{F_{\mathrm{I}}'} - \overline{f_{\mathrm{I}}} + \kappa_1 C_1 - [x - (h-d)\mathrm{i}]\overline{C_1'} - \overline{c_1}\right\} = \kappa_2 F_{\mathrm{II}} - [x - (h-d)\mathrm{i}]\overline{F_{\mathrm{II}}'} - \overline{f_{\mathrm{II}}} \\ &\Gamma\left\{\kappa_1[A_k + B_k] - [x - (h-d)\mathrm{i}]\left[\overline{A_k'} + \overline{B_k'}\right] - \overline{a_k} - \overline{b_k}\right\} = \kappa_2 F_k - [x - (h-d)\mathrm{i}]\overline{F_k'} - \overline{f_k} \\ &\Gamma\left\{\kappa_1[C_{k+1} + D_k] - [x - (h-d)\mathrm{i}]\left[\overline{C_{k+1}'} + \overline{D_k'}\right] - \overline{c_{k+1}} - \overline{d_k}\right\} = \kappa_2 G_k - [x - (h-d)\mathrm{i}]\overline{G_k'} - \overline{g_k} \end{aligned} \tag{11}$$

Here, $\Gamma = \mu_2/\mu_1$. Noting that, at the interface $z = x - (h - d)\mathrm{i}$, using this condition the two sides in the equations of Eq. (11) can be rearranged into the forms of harmonic functions. At the same time, by conjugating the two sides of the expressions of displacements, one can get:

$$\begin{aligned} &A_k' + \overline{A_k'} + [z+2(h-d)\mathrm{i}]A_k'' + a_k' - \left[F_k' + \overline{F_k'} + [z+2(h-d)i]F_k'' + f_k'\right] \\ &\qquad = -\left[B_k' + \overline{B_k'} + [z+2(h-d)\mathrm{i}]B_k'' + b_k'\right] \\ &\Gamma\left\{\kappa_1\bar{A}_k - [z+2(h-d)\mathrm{i}]A_k' - a_k\right\} - \left[\kappa_2\bar{F}_k - [z+2(h-d)\mathrm{i}]F_k' - f_k\right] \\ &\qquad = -\Gamma\left\{\kappa_1\bar{B}_k - [z+2(h-d)\mathrm{i}]B_k' - b_k\right\} \end{aligned} \tag{12}$$

$$\begin{aligned} &F_{\mathrm{I}}' + \overline{F_{\mathrm{I}}'} + [z+2(h-d)\mathrm{i}]F_{\mathrm{I}} + f_{\mathrm{I}}' - \left\{F_{\mathrm{II}}' + \overline{F_{\mathrm{II}}'} + [z+2(h-d)\mathrm{i}]F_{\mathrm{II}} + f_{\mathrm{II}}'\right\} \\ &\qquad = -\left[C_1' + \overline{C_1'} + [z+2(h-d)\mathrm{i}]C_1'' + c_1'\right] \\ &\Gamma\left\{\kappa_1\bar{F}_{\mathrm{I}} - [z+2(h-d)\mathrm{i}]F_{\mathrm{I}}' - f_{\mathrm{I}}\right\} - \left\{\kappa_2\bar{F}_{\mathrm{II}} - [z+2(h-d)\mathrm{i}]F_{\mathrm{II}}' - f_{\mathrm{II}}'\right\} \\ &\qquad = -\Gamma\left[\kappa_1\overline{C_1'} - [z+2(h-d)\mathrm{i}]C_1' - c_1\right] \end{aligned} \tag{13}$$

$$\begin{aligned} &D_k' + \overline{D_k'} + [z+2(h-d)\mathrm{i}]D_k'' + d_k' - \left\{G_k' + \overline{G_k'} + [z+2(h-d)\mathrm{i}]G_k'' + g_k'\right\} \\ &\qquad = -\left\{C_{k+1}' + \overline{C_{k+1}'} + [z+2(h-d)\mathrm{i}]C_{k+1}'' + c_{k+1}'\right\} \\ &\Gamma\left\{\kappa_1\bar{D}_k - [z+2(h-d)\mathrm{i}]D_k' - d_k\right\} - \left\{\kappa_2\bar{G}_k - [z+2(h-d)\mathrm{i}]G_k' - g_k\right\} \\ &\qquad = -\Gamma\left\{\kappa_1\bar{C}_{k+1}' - [z+2(h-d)\mathrm{i}]C_{k+1}' - c_{k+1}\right\} \end{aligned} \tag{14}$$

Based on the interchange theorem, at the interface, we have:

$$\frac{\partial L}{\partial z_k} = \frac{\partial R}{\partial \overline{\varsigma_k}}, \quad \frac{\partial L}{\partial \overline{z_k}} = \frac{\partial R}{\partial \varsigma_k}, \quad \frac{\partial L}{\partial \xi_{k+1}} = \frac{\partial R}{\partial \overline{\eta_k}}, \quad \frac{\partial L}{\partial \overline{\xi_{k+1}}} = \frac{\partial R}{\partial \eta_k} \tag{15}$$

and

$$\frac{\partial L}{\partial \xi_1} = \frac{\partial R}{\partial \bar{z}}, \quad \frac{\partial L}{\partial \overline{\xi_1}} = \frac{\partial R}{\partial z} \tag{16}$$

where, L and R represent the left- and right-hand side of a equation like $L(x,y) = R(x,y)$, respectively. Using Eqs. (15) and (16), from Eqs. (12)–(14) one can get:

$$\begin{cases} F_k = \dfrac{\Gamma(1+\kappa_1)}{\Gamma+\kappa_2}A_k \\ f_k = \left[\dfrac{\Gamma(\kappa_1+1)}{\Gamma\kappa_1+1} - \dfrac{\Gamma(\kappa_1+1)}{\Gamma+\kappa_2}\right][z+2(h-d)\mathrm{i}]A_k' + \dfrac{\Gamma(\kappa_1+1)}{\Gamma\kappa_1+1}a_k \end{cases} \tag{17a}$$

$$\begin{cases} B_k = \dfrac{\Gamma-1}{\Gamma\kappa_1+1}\left[z\overline{A_k'} + \overline{a}_k\right] \\ b_k = \dfrac{\Gamma\kappa_1-\kappa_2}{\Gamma_1+\kappa_2}\bar{A}_k + \dfrac{1-\Gamma}{\Gamma\kappa_1+1}\left\{z[z+2(h-d)\mathrm{i}]\overline{A_k''} + [z+2(h-d)\mathrm{i}]\left(\overline{A_k'} + \overline{a_k'}\right)\right\} \end{cases} \tag{17b}$$

$$\begin{cases} F_{\mathrm{II}} = \dfrac{\Gamma(1+\kappa_1)}{\Gamma+\kappa_2}F_{\mathrm{I}} \\ f_{\mathrm{II}} = \left[\dfrac{\Gamma(\kappa_1+1)}{\Gamma\kappa_1+1} - \dfrac{\Gamma(\kappa_1+1)}{\Gamma+\kappa_2}\right][z+2(h-d)\mathrm{i}]F_{\mathrm{I}}' + \dfrac{\Gamma(\kappa_1+1)}{\Gamma\kappa_1+1}f_{\mathrm{I}} \end{cases} \tag{17c}$$

$$\begin{cases} C_1 = \dfrac{\Gamma - 1}{\Gamma\kappa_1 + 1}\left[z\overline{F_{\mathrm{I}}'} + \bar{f}_{\mathrm{I}}\right] \\ c_1 = \dfrac{\Gamma\kappa_1 - \kappa_2}{\Gamma_1 + \kappa_2}\bar{F}_{\mathrm{I}} + \dfrac{1 - \Gamma}{\Gamma\kappa_1 + 1}\left\{z[z + 2(h-d)\mathrm{i}]\overline{F_{\mathrm{I}}''} + [z + 2(h-d)\mathrm{i}]\left(\overline{F}_I' + \overline{f}_{\mathrm{I}}'\right)\right\} \end{cases} \tag{17d}$$

$$\begin{cases} G_k = \dfrac{\Gamma(1 + \kappa_1)}{\Gamma + \kappa_2}D_k \\ g_k = \left[\dfrac{\Gamma(\kappa_1 + 1)}{\Gamma\kappa_1 + 1} - \dfrac{\Gamma(\kappa_1 + 1)}{\Gamma + \kappa_2}\right][z + 2(h-d)\mathrm{i}]D_k' + \dfrac{\Gamma(\kappa_1 + 1)}{\Gamma\kappa_1 + 1}d_k \end{cases} \tag{17e}$$

$$\begin{cases} C_{k+1} = \dfrac{\Gamma - 1}{\Gamma\kappa_1 + 1}\left[z\overline{D_k'} + \overline{d}_k\right] \\ c_{k+1} = \dfrac{\Gamma\kappa_1 - \kappa_2}{\Gamma_1 + \kappa_2}\bar{D}_k + \dfrac{1 - \Gamma}{\Gamma\kappa_1 + 1}\left\{z[z + 2(h-d)\mathrm{i}]\overline{D_k''} + [z + 2(h-d)\mathrm{i}]\left(\overline{D_k'} + \overline{d}_k'\right)\right\} \end{cases} \tag{17f}$$

When A_k and a_k are known, other functions can be obtained. Using Dundur's parameters, i.e.:

$$\alpha = \frac{\mu_1(\kappa_2 + 1) - \mu_2(\kappa_1 + 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)}, \quad \beta = \frac{\mu_1(\kappa_2 - 1) - \mu_2(\kappa_1 - 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)} \tag{18}$$

the equations of recurrence can be rewritten as:

$$\begin{cases} F_k = \dfrac{1 - \alpha}{1 + \beta}A_k \\ f_k = \dfrac{2(1 - \alpha)\beta}{1 - \beta^2}[z + 2(h-d)\mathrm{i}]A_k' + \dfrac{1 - \alpha}{1 - \beta}a_k \end{cases} \tag{19a}$$

$$\begin{cases} B_k = \dfrac{\beta - \alpha}{1 - \beta}\left[z\overline{A}_k' + \overline{a}_k\right] \\ b_k = -\dfrac{\alpha + \beta}{1 + \beta}\overline{A}_k + \dfrac{\alpha - \beta}{1 - \beta}\left\{z[z + 2(h-d)\mathrm{i}]\overline{A_k''} + [z + 2(h-d)\mathrm{i}]\left(\overline{A_k'} + \overline{a_k'}\right)\right\} \end{cases} \tag{19b}$$

$$\begin{cases} F_{\mathrm{II}} = \dfrac{1 - \alpha}{1 + \beta}F_{\mathrm{I}} \\ f_{\mathrm{II}} = \dfrac{2(1 - \alpha)\beta}{1 - \beta^2}[z + 2(h-d)\mathrm{i}]F_{\mathrm{I}}' + \dfrac{1 - \alpha}{1 - \beta}f_{\mathrm{I}} \end{cases} \tag{19c}$$

$$\begin{cases} C_1 = \dfrac{\beta - \alpha}{1 - \beta}\left[z\overline{F_{\mathrm{I}}'} + \overline{f_{\mathrm{I}}}\right] \\ c_1 = -\dfrac{\alpha + \beta}{1 + \beta}\bar{F}_{\mathrm{I}} + \dfrac{\alpha - \beta}{1 - \beta}\left\{z[z + 2(h-d)\mathrm{i}]\overline{F_{\mathrm{I}}''} + [z + 2(h-d)\mathrm{i}]\left(\overline{F_{\mathrm{I}}'} + \overline{f_{\mathrm{I}}'}\right)\right\} \end{cases} \tag{19d}$$

$$\begin{cases} G_k = \dfrac{1 - \alpha}{1 + \beta}D_k \\ g_k = \dfrac{2(1 - \alpha)\beta}{1 - \beta^2}[z + 2(h-d)\mathrm{i}]D_k' + \dfrac{1 - \alpha}{1 - \beta}d_k \end{cases} \tag{19e}$$

$$\begin{cases} C_{k+1} = \dfrac{\beta - \alpha}{1 - \beta}\left[z\overline{D_k'} + \overline{d}_k\right] \\ c_{k+1} = -\dfrac{\alpha + \beta}{1 + \beta}\overline{D}_k + \dfrac{\alpha - \beta}{1 - \beta}\left\{z[z + 2(h-d)\mathrm{i}]\overline{D_k''} + [z + 2(h-d)\mathrm{i}]\left(\overline{D_k'} + \overline{d_k'}\right)\right\} \end{cases} \tag{19f}$$

Note that the above recurrence relations are derived via the continuity condition at the interface. In the next section, traction-free condition at the surface will be discussed and the other stress functions recurrence relationships will be derived.

3.3. Stress functions recurrence for image points symmetric to surface

At the free surface, we have:

$$z = x + d\mathrm{i}, \quad z_{k+1} = \bar{\varsigma}_k, \quad \xi_k = \bar{\eta}_k \tag{20}$$

Applying the following interchange theorem

$$\frac{\partial L}{\partial z_{k+1}} = \frac{\partial R}{\partial \overline{\varsigma_k}}, \quad \frac{\partial L}{\partial \overline{z_{k+1}}} = \frac{\partial R}{\partial \varsigma_k}, \quad \frac{\partial L}{\partial \xi_k} = \frac{\partial R}{\partial \overline{\eta_k}}, \quad \frac{\partial L}{\partial \overline{\xi_k}} = \frac{\partial R}{\partial \eta_k} \tag{21}$$

and

$$\frac{\partial L}{\partial z_1} = \frac{\partial R}{\partial \bar{z}}, \quad \frac{\partial L}{\partial \overline{z_1}} = \frac{\partial R}{\partial z} \tag{22}$$

substituting Eqs. (8) and (9) into Eq. (3), considering the corresponding relationships of the image points, the following expressions can be obtained:

$$\begin{aligned} &F_{\mathrm{I}}' + \overline{F_{\mathrm{I}}'} + (x - d\mathrm{i})F_{\mathrm{I}}'' + f_{\mathrm{I}}' + A_1' + \overline{A_1'} + (x - di)A_1'' + a_1' = 0 \\ &A_{k+1}' + B_k' + \overline{A_{k+1}'} + \overline{B_k'} + (x - d\mathrm{i})[A_{k+1}'' + B_k''] + a_{k+1}' + b_k' = 0 \\ &C_k' + D_k' + \overline{C_k'} + \overline{D_k'} + (x - d\mathrm{i})[C_k'' + D_k''] + c_k' + d_k' = 0 \end{aligned} \tag{23}$$

Applying $z = x + d\mathrm{i}$, the above expressions can be rewritten as the following harmonic functions:

$$\begin{aligned} &A_1' + \overline{A_1'} + (z - 2d\mathrm{i})A_1'' + a_1' = -\left[F_{\mathrm{I}}' + \overline{F_{\mathrm{I}}'} + (z - 2d\mathrm{i})F_{\mathrm{I}}'' + f_{\mathrm{I}}'\right] \\ &A_{k+1}' + \overline{A_{k+1}'} + (z - 2d\mathrm{i})A_{k+1}'' + a_{k+1}' = -\left[B_k' + \overline{B_k'} + (z - 2d\mathrm{i})B_k'' + b_k'\right] \\ &D_k' + \overline{D_k'} + (z - 2d\mathrm{i})D_k'' + d_k' = -\left[C_k' + \overline{C_k'} + (z - 2d\mathrm{i})C_k'' + c_k'\right] \end{aligned} \tag{24}$$

Employing the relationships shown in Eq. (21), the recurrence relationships of the stress functions can be obtained as:

$$\begin{cases} A_{k+1} = -z\bar{B}_k' - \bar{b}_k' \\ a_{k+1} = (z - 2d\mathrm{i})\bar{B}_k' - \bar{B}_k + z(z - 2d\mathrm{i})\bar{B}_k'' + (z - 2d\mathrm{i})\bar{b}_k' \end{cases} \tag{25a}$$

$$\begin{cases} A_1 = -z\bar{F}_{\mathrm{I}}' - \bar{f}_{\mathrm{I}} \\ a_1 = (z - 2d\mathrm{i})\bar{F}_{\mathrm{I}}' - \bar{F}_{\mathrm{I}} + z(z - 2d\mathrm{i})\bar{F}_{\mathrm{I}}'' + (z - 2d\mathrm{i})\bar{f}' \end{cases} \tag{25b}$$

$$\begin{cases} D_k = -z\bar{C}_k' - \bar{c}_k \\ d_k = (z - 2d\mathrm{i})\bar{C}_k' - \bar{C}_k + z(z - 2d\mathrm{i})\bar{C}_k'' + (z - 2d\mathrm{i})\bar{c}' \end{cases} \tag{25c}$$

Let us define:

$$F_{\mathrm{I}}(z) = N\log z, \quad f_I(z) = -\kappa_1\bar{N}\log z, \quad N = -\frac{P_x + \mathrm{i}P_y}{2\pi(\kappa_1 + 1)} \tag{26}$$

Actually, F_{I}, f_{I} are the stress functions for the unbounded homogeneous plane under a concentrated force. Substituting Eq. (26) into Eqs. (19d) and (25b) the functions C_1, c_1, A_1 and a_1 can be obtained. Based on these functions, the other stress functions terms can be derived using Eqs. (19a), (19b), (19c), (19e), (19f), (25a) and (25c). This method is simple and straightforward but a little inconvenient. The more effective way is to set each stress function as a polynomial form and then using the recurrence relationships to derive the general formulas (see Appendix A). Actually, from the final recurrence relationships in polynomial forms the coefficients can be solved using mathematical software like MATLAB, MATHAMATICA or MAPLE.

4. Comparisons of the theoretical solutions to the results of FEM

In order to show the validity of the above theories, FEM computations also have been carried out. The FEM model adopted here is shown in Fig. 2. Since the film is very thin and the actual theoretical problem is about half-infinite plane, the dimensions of the substrate must be very large compared to the upper thin layer. The finite element mesh is shown in Fig. 3. The materials constants are given in Table 1. Here, the

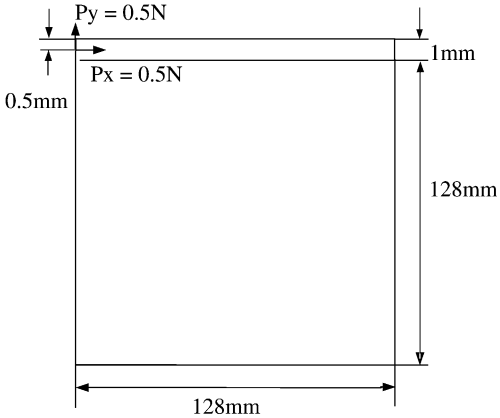

Fig. 2. FEM analysis model.

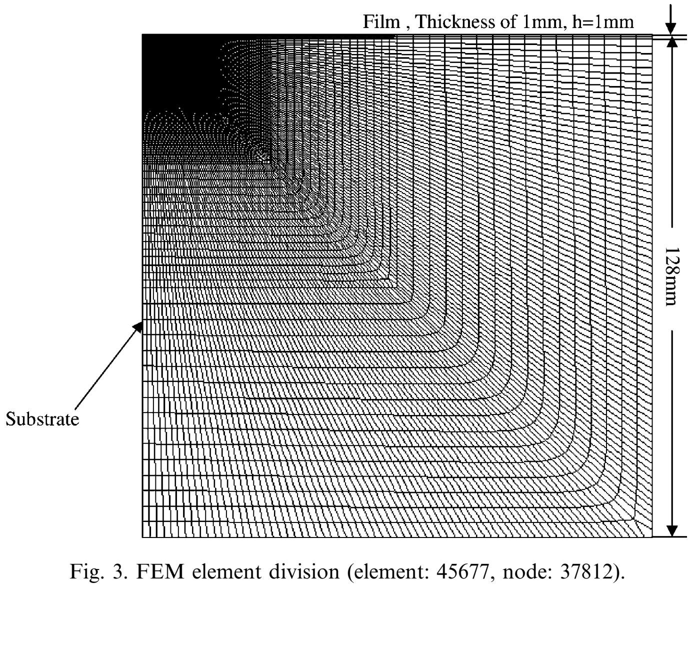

Fig. 3. FEM element division (element: 45677, node: 37812).

Table 1
Materials constant

Material	I	II
Young's modulus E (Gpa)	546	206
Poisson's ratio ν	0.3	0.3

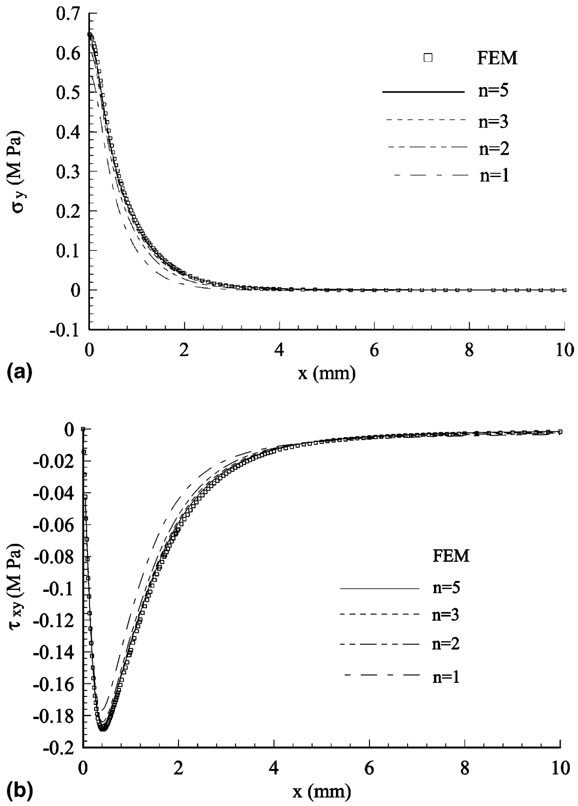

Fig. 4. (a) Normal stress σ_y for $P_x = 0$, $P_y = 1$ N/mm; (b) shear stress τ_{xy} for $P_x = 0$, $P_y = 1$ N/mm.

distance between the load point and the free surface is 0.5 mm. Generally, the stresses along the interface are most important. Thus, interface stresses comparisons of four cases between the theoretical and numerical results from FEM are performed, shown in Figs. 4 and 5.

In the legends of the figures, $n = k$ denotes the order of image point. From comparisons to the results of FEM, it can be found that the stress increments decrease rapidly as the order of the image points increases. And the stresses resulting from superimposing five image points have enough accuracy for the materials combination in the Table 1. These imply that the convergence rate of the theoretical solutions is very fast.

5. Effects of combination of materials

In the Section 4, the theoretical and numerical results are compared. We can see that the stress increments corresponding to the higher order image points decrease rapidly. Therefore for practical problem, a

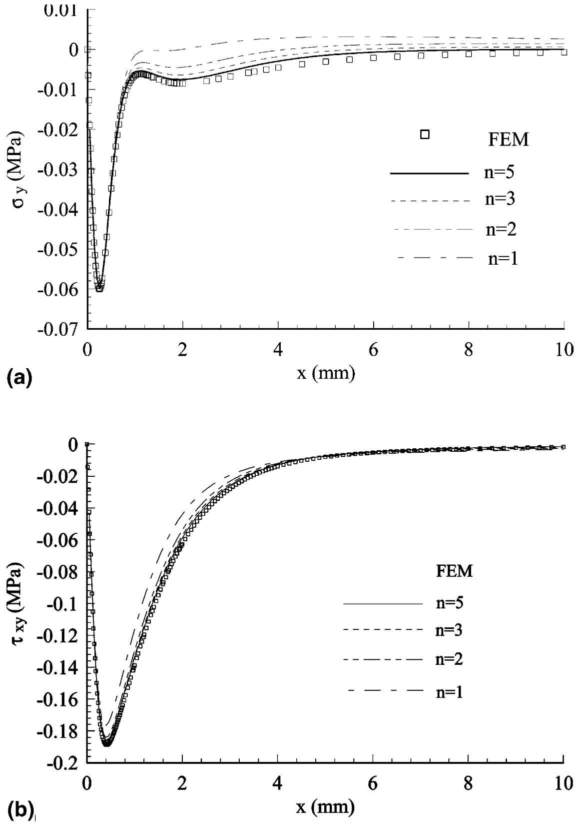

Fig. 5. (a) Normal stress σ_y for $P_x = 1$ N/mm, $P_y = 0$; (b) shear stress τ_{xy} for $P_x = 1$ N/mm, $P_y = 0$.

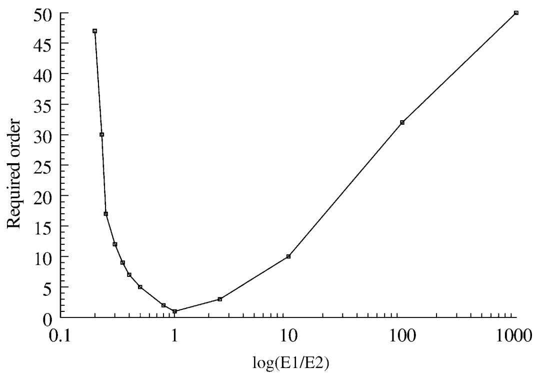

Fig. 6. Required order for different Young's modulus.

few terms of stress functions are sufficient to ensure the accuracy. In this section, the effects of the difference between the material properties will be discussed. Here, assume Poisson's ratios as $\nu_1 = \nu_2 = 0.3$. The required orders for different Young's modulus are calculated and the results are shown in Fig. 6. From the curve, it can be seen that when the mismatch between the Young's modulus is bigger, the more image points are needed to get satisfactory accuracy.

6. Conclusions

Through the image method, the theoretical solutions are obtained for modified material subjected to a concentrated force in the upper layer, the conclusions are:

1. Theoretical stress functions corresponding to each image point, from the lower order to higher, exhibit recurrence relations, and the final solution is an infinite series.
2. The stresses increments decrease rapidly when the order of image points is higher. It implies that the convergence rate of the solutions is very rapid.
3. The distributions of stresses at the interface are most important. Generally, for most practical materials combination, only five image points are needed to get satisfactory accuracy.
4. If the material properties between the upper layer and the substrate are very different, more image points are needed to ensure the accuracy of the theoretical solution.

Appendix A. Recurrence relationships of stress functions

Dundur's parameters:

$$\alpha = \frac{\mu_1(\kappa_2+1)-\mu_2(\kappa_1+1)}{\mu_1(\kappa_2+1)+\mu_2(\kappa_1+1)}, \quad \beta = \frac{\mu_1(\kappa_2-1)-\mu_2(\kappa_1-1)}{\mu_1(\kappa_2+1)+\mu_2(\kappa_1+1)} \tag{A.1}$$

Let us define:

$$m_1 = \frac{\beta-\alpha}{1-\beta}, \quad m_2 = \frac{\alpha+\beta}{1+\beta} \tag{A.2}$$

The expressions of recurrence in Eqs. (19a)–(19f), (25a)–(25c) can be rearranged as:

$$\begin{aligned} F_k &= (1-m_2)A_k \\ f_k &= (m_1+m_2)(z_k+2kh\mathrm{i})A_k' + (m_1+1)a_k \end{aligned} \tag{A.3}$$

$$\begin{aligned} B_k &= m_1\lfloor(\varsigma_k-2kh\mathrm{i})\bar{A}_k' + \bar{a}_k\rfloor \\ b_k &= -m_2\bar{A}_k - m_1\left\{(\varsigma_k-2kh\mathrm{i})[\varsigma_k-2kh\mathrm{i}+2(h-d)\mathrm{i}]\bar{A}_k' + [\varsigma_k-2kh\mathrm{i}+2(h-d)\mathrm{i}]\left(\bar{A}_k'+\bar{a}_k'\right)\right\} \end{aligned} \tag{A.4}$$

$$\begin{aligned} A_{k+1} &= -z\bar{B}_k' - \bar{b}_k = -[z_{k+1}+(2d+2kh)\mathrm{i}]\bar{B}_k' - \bar{b}_k \\ a_{k+1} &= (z-2d\mathrm{i})\bar{B}_k' - \bar{B}_k + z(z-2d\mathrm{i})\bar{B}_k'' + (z-2d\mathrm{i})\bar{b}_k' \\ &= (z_{k+1}+2kh\mathrm{i})\bar{B}_k' - \bar{B}_k + (z_{k+1}+2kh\mathrm{i})[z_{k+1}+(2d+2kh)\mathrm{i}]\bar{B}_k'' + (z_{k+1}+2kh\mathrm{i})\bar{b}_k' \end{aligned} \tag{A.5}$$

$$\begin{cases} F_{\mathrm{II}} = (1-m_2)F_{\mathrm{I}} \\ f_{\mathrm{II}} = (m_1+m_2)[z+2(h-d)\mathrm{i}]F_{\mathrm{I}}' + (m_1+1)f_{\mathrm{I}} \end{cases} \tag{A.6}$$

$$\begin{aligned} &\xi_1 = z+2(h-d)\mathrm{i}, \quad z = \xi_1 - 2(h-d)\mathrm{i} \quad \xi_1 = \bar{z} \\ &\begin{cases} C_1 = m_1\left\{[\xi_1-2(h-d)\mathrm{i}]\bar{F}_{\mathrm{I}}' + \bar{f}_{\mathrm{I}}\right\} \\ c_1 = -m_2\bar{F}_{\mathrm{I}} - m_1\left\{\xi_1[\xi_1-2(h-d)\mathrm{i}]\bar{F}_{\mathrm{I}}'' + \xi_1\left(\bar{F}_{\mathrm{I}}'+\bar{f}_{\mathrm{I}}'\right)\right\} \end{cases} \end{aligned} \tag{A.7}$$

$$\begin{cases} G_k = (1-m_2)D_k \\ g_k = (m_1+m_2)[\eta_k+2kh\mathrm{i}+2(h-d)\mathrm{i}]D_k' + (m_1+1)d_k \end{cases} \tag{A.8}$$

$$\xi_k = z + (2kh - 2d)\mathrm{i}, \quad z = \xi_k - (2kh - 2d)\mathrm{i}, \quad \xi_{k+1} = \bar{\eta}_k$$

$$\begin{cases} C_{k+1} = m_1\left\{[\xi_{k+1} - (2kh - 2d)\mathrm{i}]\bar{D}'_k + \bar{d}_k\right\} \\ c_{k+1} = -m_2\bar{D}_k - m_1\left\{\begin{array}{l}[\xi_{k+1} - (2kh - 2d)\mathrm{i}][\xi_{k+1} - (2kh - 2d)\mathrm{i} + 2(h-d)\mathrm{i}]\bar{D}''_k \\ +[\xi_{k+1} - (2kh - 2d)\mathrm{i} + 2(h-d)\mathrm{i}]\left(\bar{D}'_k + \bar{d}'_k\right)\end{array}\right\} \end{cases} \tag{A.9}$$

$$\eta_k = z - 2kh\mathrm{i}, \quad z = \eta_k + 2kh\mathrm{i}, \quad \eta_k = \bar{\xi}_k$$

$$\begin{cases} D_k = -(\eta_k + 2kh\mathrm{i})\bar{C}'_k - \bar{c}_k \\ d_k = (\eta_k + 2kh\mathrm{i} - 2\mathrm{i}d)\bar{C}'_k - \bar{C}_k + (\eta_k + 2kh\mathrm{i})(\eta_k + 2kh\mathrm{i} - 2\mathrm{i}d)\bar{C}''_k + (\eta_k + 2k\mathrm{i}h - 2\mathrm{i}d)\bar{c}'_k \end{cases} \tag{A.10}$$

A.1. Polynomial forms of stress functions

With the following definition:

$$A_k = u_0 + u_1 \log z_k + \sum_{j=1}^{N} \frac{u_{j+1}}{z_k^j}, \quad a_k = v_0 + v_1 \log z_k + \sum_{j=1}^{N+1} \frac{v_{j+1}}{z_k^j} \tag{A.11}$$

by substituting into Eq. (A.4), we obtain

$$B_k = m_1\left\{\overline{(u_1 + v_0)} + \bar{v}_1 \log \varsigma_k + \frac{\overline{(v_2 - u_2)} - 2\mathrm{i}kh\bar{u}_1}{\varsigma_k} + \sum_{j=2}^{N} \frac{\overline{(v_{j+1} - ju_{j+1})} + 2\mathrm{i}(j-1)kh\bar{u}_j}{\varsigma_k^j} \right.$$
$$\left. + \frac{2\mathrm{i}Nkh\bar{u}_{N+1} + \bar{v}_{N+2}}{\varsigma_k^{N+1}}\right\} \tag{A.12}$$

$$b_k = -\left(m_2\bar{u}_0 + m_1\bar{v}_1\right) - m_2\bar{u}_1 \ln \varsigma_k + \frac{2m_1\mathrm{i}(kh - h + d)\bar{v}_1 - 2\mathrm{i}khm_1\bar{u}_1 + m_1\bar{v}_2 - (m_1 + m_2)\bar{u}_2}{\varsigma_k}$$
$$+ \frac{2m_1\bar{v}_3 - (4m_1 + m_2)\bar{u}_3 - 4m_1(kh - h + d)kh\bar{u}_1 - 2\mathrm{i}m_1kh\overline{(v_2 - 3u_2)} + 2\mathrm{i}m_1(h-d)\overline{(v_2 - u_2)}}{\varsigma_k^2}$$
$$+ \sum_{j=3}^{N} \frac{\left\{\begin{array}{l} jm_1\bar{v}_{j+1} - (j^2m_1 + m_2)\bar{u}_{j+1} - 2\mathrm{i}m_1(kh - h + d)\left[(j-1)\bar{v}_j - (j-1)^2\bar{u}_j\right] \\ +2\mathrm{i}m_1j(j-1)kh\bar{u}_j + 4m_1(j-1)(j-2)(kh - h + d)kh\bar{u}_{j-1}\end{array}\right\}}{\varsigma_k^j}$$
$$+ \frac{m_1\left\{\begin{array}{l}(N+1)\bar{v}_{N+2} + 4N(N-1)(kh - h + d)kh\bar{u}_N \\ -2\mathrm{i}N(kh - h + d)\left[\bar{v}_{N+1} - N\bar{u}_{N+1}\right]\end{array}\right\} + 2\mathrm{i}N(N+1)kh\bar{u}_{N+1}}{\varsigma_k^{N+1}}$$
$$+ \frac{m_1\left[4N(N+1)(kh - h + d)kh\bar{u}_{N+1} - 2\mathrm{i}(N+1)(kh - h + d)\bar{v}_{N+2}\right]}{\varsigma_k^{N+2}} \tag{A.13}$$

Substitute Eq. (A.11) into Eq. (A.3), we get

$$f_k = (m_1 + m_2)u_1 + (m_1 + 1)v_0 + (m_1 + 1)v_1 \ln(z_k) + \frac{1}{z_k}[2ikh(m_1 + m_2)u_1 - (m_1 + m_2)u_2 + (m_1 + 1)v_2] - \sum_{j=2}^{N} \frac{1}{z_k^j}\left[j(m_1 + m_2)u_{j+1} - (m_1 + 1)v_{j+1} + 2ikh(j-1)(m_1 + m_2)u_j\right] + \frac{1}{z_k^{N+1}}[(m_1 + 1)v_{N+2} - 2ikhN(m_1 + m_2)u_{N+1}] \tag{A.14}$$

Similarly, set:

$$B_k = p_0 + p_1 \ln \varsigma_k + \sum_{j=1}^{N} \frac{p_{j+1}}{\varsigma_k^j} \quad b_k = q_0 + q_1 \log \varsigma_k + \sum_{j=1}^{N+1} \frac{q_{j+1}}{\varsigma_k^j} \tag{A.15}$$

then substitute the above equation into Eq. (A.5)

$$A_{k+1} = -\overline{(p_1 + q_0)} - \bar{q}_1 \ln(z_{k+1}) + \frac{\overline{(p_2 - q_2)} - 2i(kh + d)\bar{p}_1}{z_{k+1}} + \sum_{j=2}^{N} \frac{\overline{(jp_{j+1} - q_{j+1})} + 2i(j-1)(kh + d)\bar{p}_j}{z_{k+1}^j} + \frac{2iN(kh + d)\bar{p}_{N+1} - \bar{q}_{N+2}}{z_{k+1}^{N+1}} \tag{A.16}$$

$$\begin{aligned} a_{k+1} = {} & \overline{(q_1 - p_0)} - \bar{p}_1 \ln z_{k+1} - \frac{\bar{q}_2 + 2i\left[(kh + d)\bar{p}_1 - kh\bar{q}_1\right]}{z_{k+1}} \\ & + \frac{4(kh + d)kh\bar{p}_1 - 2\bar{q}_3 + 3\bar{p}_3 + 2i\left[(3kh + 2d)\bar{p}_2 - kh\bar{q}_2\right]}{z_{k+1}^2} \\ & + \sum_{j=3}^{N} \frac{\left\{\begin{array}{l} 2i(j-1)\left[(2jkh + jd - kh)\bar{p}_j - kh\bar{q}_j\right] - (1 - j^2)\bar{p}_{j+1} \\ -j\bar{q}_{j+1} - 4kh(kh + d)(j-1)(j-2)\bar{p}_{j-1} \end{array}\right\}}{z_{k+1}^j} \\ & + \frac{2iN[(2N + 1)kh + (N + 1)d]\bar{p}_{N+1} - 4khN(N-1)(kh + d)\bar{p}_N - (N + 1)\bar{q}_{N+2} - 2ikhN\bar{q}_{N+1}}{z_{k+1}^{N+1}} \\ & - \frac{2ikh(N + 1)\bar{q}_{N+2} + 4N(N + 1)kh(kh + d)\bar{p}_{N+1}}{z_{k+1}^{N+2}} \end{aligned} \tag{A.17}$$

If set:

$$D_k = u_0 + u_1 \log \eta_k + \sum_{j=1}^{N} \frac{u_{j+1}}{\eta_k^j}, \quad d_k = v_0 + v_1 \log \eta_k + \sum_{j=1}^{N+1} \frac{v_{j+1}}{\eta_k^j} \tag{A.18}$$

then

$$C_{k+1} = m_1\{\overline{(u_1+v_0)} + \bar{v}_1 \log \xi_{k+1} + \frac{\overline{(v_2-u_2)} - 2\mathrm{i}(kh-d)\bar{u}_1}{\xi_{k+1}} + \sum_{j=2}^{N} \frac{\overline{(v_{j+1}-ju_{j+1})} + 2\mathrm{i}(j-1)(kh-d)\bar{u}_j}{\xi_{k+1}^j} + \frac{2\mathrm{i}N(kh-d)\bar{u}_{N+1} + \bar{v}_{N+2}}{\xi_{k+1}^{N+1}}\} \tag{A.19}$$

$$\begin{aligned} c_{k+1} = &-\left(m_2\bar{u}_0 + m_1\bar{v}_1\right) - m_2\bar{u}_1 \ln \xi_{k+1} \\ &+ \frac{2m_1\mathrm{i}(kh-d)\overline{(v_1-u_1)} + m_1\bar{v}_2 - 2\mathrm{i}(h-d)m_1\bar{v}_1 - (m_1+m_2)\bar{u}_2}{\xi_{k+1}} \\ &+ \frac{2m_1\bar{v}_3 - (4m_1+m_2)\bar{u}_3 - 4m_1(kh-d)(kh-h)\bar{u}_1 - 2\mathrm{i}m_1(kh-h)\overline{(v_2-3u_2)}4\mathrm{i}(h-d)m_1\bar{u}_2}{\xi_{k+1}^2} \\ &+ \sum_{j=3}^{N} \frac{\left\{\begin{array}{l} jm_1\bar{v}_{j+1} - (j^2m_1+m_2)\bar{u}_{j+1} - 2\mathrm{i}m_1(kh-h)\left[(j-1)\bar{v}_j - (j-1)j\bar{u}_j\right] \\ +2\mathrm{i}jm_1(j-1)(kh-h)\bar{u}_j + 4m_1(j-1)(j-2)(kh-d)(kh-h)\bar{u}_{j-1} \end{array}\right\}}{\xi_{k+1}^j} \\ &+ \frac{m_1\left\{\begin{array}{l} (N+1)\bar{v}_{N+2} + 4N(N-1)(kh-d)(kh-h)\bar{u}_N \\ -2\mathrm{i}N(kh-h)\left[\bar{v}_{N+1} - N\bar{u}_{N+1}\right] \end{array}\right\}2\mathrm{i}N(N+1)(kh-d)\bar{u}_{N+1}}{\xi_{k+1}^{N+1}} \\ &+ \frac{m_1\left[4N(N+1)(kh-d)(kh-h)\bar{u}_{N+1} - 2\mathrm{i}(N+1)(kh-h)\bar{v}_{N+2}\right]}{\xi_{k+1}^{N+2}} \end{aligned} \tag{A.20}$$

$$\begin{aligned} g_k = &(m_1+m_2)u_1 + (m_1+1)v_0 + (m_1+1)v_1 \ln(\eta_k) + \frac{1}{\eta_k}[2\mathrm{i}(kh+h-d)(m_1+m_2)u_1 - (m_1+m_2)u_2 \\ &+ (m_1+1)v_2] - \sum_{j=2}^{N} \frac{1}{\eta_k^j}[\mathrm{i}(m_1+m_2)u_{j+1} - (m_1+1)v_{j+1} + 2\mathrm{i}(kh+h-d)(j-1)(m_1+m_2)u_j] \\ &+ \frac{1}{\eta_k^{N+1}}[(m_1+1)v_{N+2} - 2\mathrm{i}(kh+h-d)N(m_1+m_2)u_{N+1}] \end{aligned} \tag{A.21}$$

Similarly, set:

$$C_k = p_0 + p_1 \ln \xi_k + \sum_{j=1}^{N} \frac{p_{j+1}}{\xi_k^j} \quad c_k = q_0 + q_1 \log \xi_k + \sum_{j=1}^{N+1} \frac{q_{j+1}}{\xi_k^j} \tag{A.22}$$

then

$$\begin{aligned} D_k = &-\overline{(p_1+q_0)} - \bar{q}_1 \ln(\eta_k) + \frac{\overline{(p_2-q_2)} - 2\mathrm{i}kh\bar{p}_1}{\eta_k} + \sum_{j=2}^{N} \frac{\overline{(jp_{j+1}-q_{j+1})} + 2\mathrm{i}(j-1)kh\bar{p}_j}{\eta_k^j} \\ &+ \frac{2\mathrm{i}Nkh\bar{p}_{N+1} - \bar{q}_{N+2}}{\eta_k^{N+1}} \end{aligned} \tag{A.23}$$

$$
\begin{aligned}
d_k = {} & \overline{(q_1 - p_0)} - \bar{p}_1 \ln \eta_k - \frac{\bar{q}_2 + 2\mathrm{i}\left[kh\bar{p}_1 - (kh-d)\bar{q}_1\right]}{\eta_k} \\
& + \frac{4(kh-d)kh\bar{p}_1 - 2\bar{q}_3 + 3\bar{p}_3 + 2\mathrm{i}\left[(3kh-d)\bar{p}_2 - (kh-d)\bar{q}_2\right]}{\eta_k^2} \\
& + \sum_{j=3}^{N} \frac{\left\{\begin{array}{l} 2\mathrm{i}(j-1)\left[(2jkh - jd - kh + d)\bar{p}_j - (kh-d)\bar{q}_j\right] - (1-j^2)\bar{p}_{j+1} \\ -j\bar{q}_{j+1} - 4kh(kh-d)(j-1)(j-2)\bar{p}_{j-1} \end{array}\right\}}{\eta_k^j} \\
& + \frac{\left\{\begin{array}{l} 2\mathrm{i}N[(N+1)(2kh-d) - (kh-d)]\bar{p}_{N+1} - 4khN(N-1)(kh-d)\bar{p}_N \\ -(N+1)\bar{q}_{N+2} - 2\mathrm{i}(kh-d)N\bar{q}_{N+1} \end{array}\right\}}{\eta_k^{N+2}} \\
& - \frac{2\mathrm{i}(kh-d)(N+1)\bar{q}_{N+2} + 4N(N+1)kh(kh-d)\bar{p}_{N+1}}{\eta_k^{N+2}} \qquad (A.24)
\end{aligned}
$$

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